



ENGINEERING MECHANICS I:

CENTROIDS

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Chapter 5

Centroids

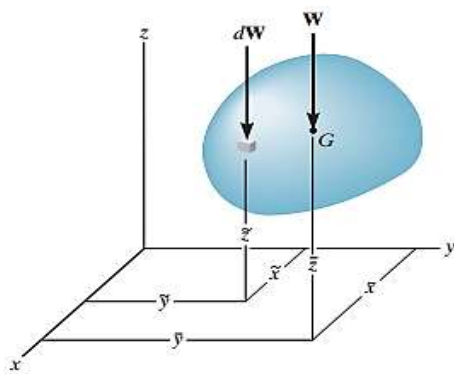
5.1. Introduction

In case of plane areas (bodies with negligible thickness) such as a triangle quadrilateral, circle etc., the total area is assumed to be concentrated at a single point and such a single point is called centroid of the plane area. The term center of gravity and centroid has the same meaning but the following differences.

1. Centre of gravity refer to bodies with mass and weight whereas, centroid refers to plane areas.
2. Centre of gravity is a point in a body through which the weight acts vertically downwards irrespective of the position, whereas the centroid is a point in a plane area such that the moment of areas about an axis through the centroid is zero.

5.2. Center of Gravity

Center of Gravity A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight dW . These weights will form a parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the **center of gravity**, G .



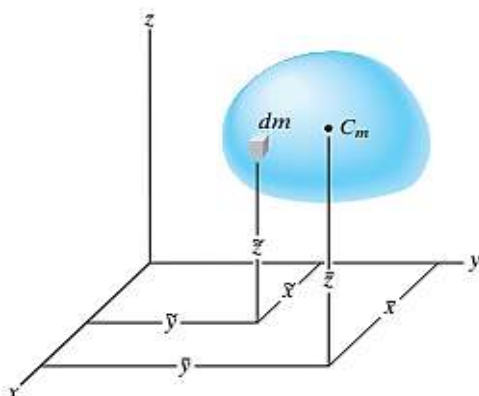
$$\bar{x} = \frac{\int \tilde{x} dW}{\int dW} \quad \bar{y} = \frac{\int \tilde{y} dW}{\int dW} \quad \bar{z} = \frac{\int \tilde{z} dW}{\int dW}$$

where

$\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the center of gravity G .

$\tilde{x}, \tilde{y}, \tilde{z}$ are the coordinates of an arbitrary particle in the body.

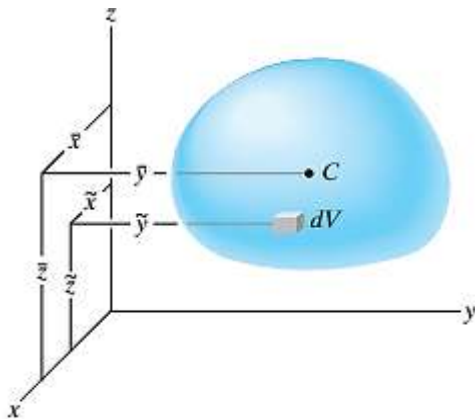
Center of Mass of a Body. In order to study the dynamic response or accelerated motion of a body, it becomes important to locate the body's center of mass C_m , figure. This location can be determined by substituting $dW = g dm$ into next equations. Provided g is constant, it cancels out, and so



$$\bar{x} = \frac{\int \tilde{x} dm}{\int dm} \quad \bar{y} = \frac{\int \tilde{y} dm}{\int dm} \quad \bar{z} = \frac{\int \tilde{z} dm}{\int dm}$$

5.3. Centroids of lines, Areas, and Volumes

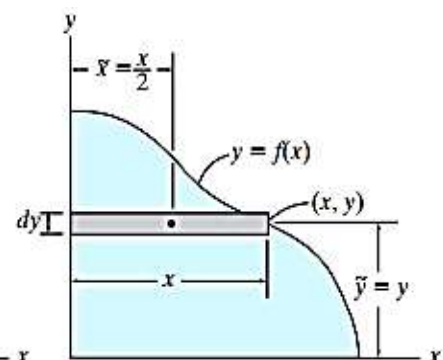
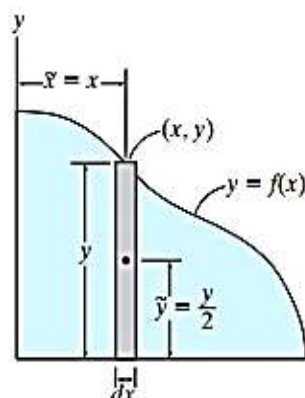
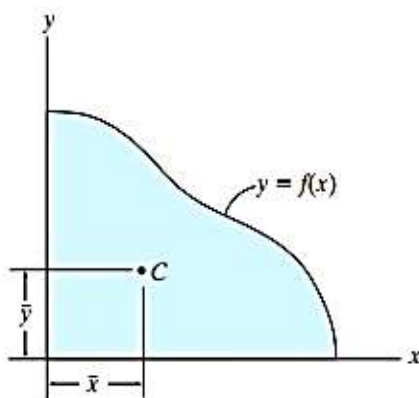
Centroid of a Volume. If the body in figure is made from a homogeneous material, then its density ρ (rho) will be constant. Therefore, a differential element of volume dV has a mass $dm = \rho dV$. Substituting this into equations and canceling out ρ , we obtain formulas that locate the centroid C or geometric center of the body; namely



$$\bar{x} = \frac{\int_V \tilde{x} dV}{\int_V dV} \quad \bar{y} = \frac{\int_V \tilde{y} dV}{\int_V dV} \quad \bar{z} = \frac{\int_V \tilde{z} dV}{\int_V dV}$$

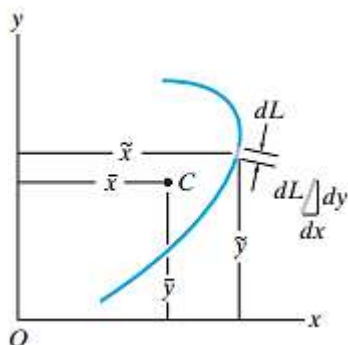
Centroid of an Area. If an area lies in the x - y plane and is bounded by the curve $y = f(x)$, as shown in figure, then its centroid will be in this plane and can be determined from integrals similar to equations, namely,

$$\bar{x} = \frac{\int_A \tilde{x} dA}{\int_A dA} \quad \bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA}$$



Centroid of a Line. If a line segment (or rod) lies within the x - y plane and it can be described by a thin curve $y = f(x)$, figure, then its centroid is determined from

$$\bar{x} = \frac{\int_L \tilde{x} dL}{\int_L dL} \quad \bar{y} = \frac{\int_L \tilde{y} dL}{\int_L dL}$$



$$\begin{aligned} dL &= \sqrt{\left(\frac{dx}{dx}\right)^2 dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2} & dL &= \sqrt{\left(\frac{dx}{dy}\right)^2 dy^2 + \left(\frac{dy}{dy}\right)^2 dy^2} \\ &= \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx & &= \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right) dy \end{aligned}$$

5.4. Determination of centroid by integrations

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

Differential Element.

- ✓ Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- ✓ For lines the element is represented by a differential line segment of length dL .
- ✓ For areas the element is generally a rectangle of area dA , having a finite length and differential width.
- ✓ For volumes the element can be a circular disk of volume dV , having a finite radius and differential thickness.
- ✓ Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

Size and Moment Arms.

- ✓ Express the length dL , area dA , or volume dV of the element in terms of the coordinates describing the curve.
- ✓ Express the moment arms x, y, z for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

Integrations.

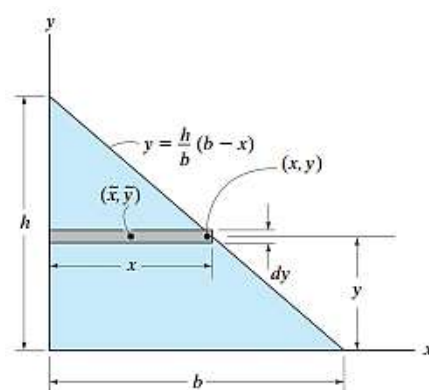
- ✓ Substitute the formulations for x, y, z and dL, dA , or dV into the appropriate equations.
- ✓ Express the function in the integrand in terms of the same variable as the differential thickness of the element.
- ✓ The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are “summed” or the integration performed, the entire region is covered.

Example 5.1.

Determine the distance y measured from the x axis to the centroid of the area of the triangle shown in figure.

Solution

Differential Element. Consider a rectangular element having a thickness dy , and located in an arbitrary position so that it intersects the boundary at (x, y)



Area and Moment Arms. The area of the element is $dA = x \, dy$ and its centroid is located a distance

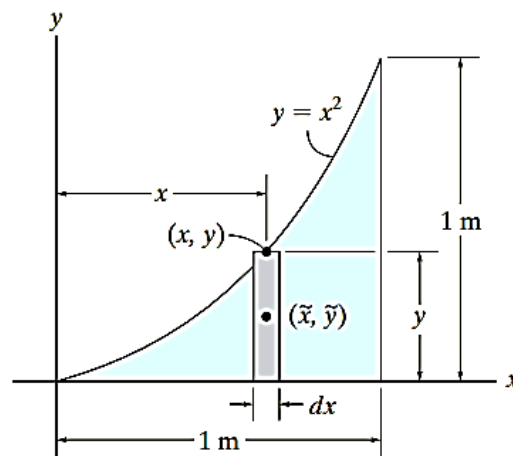
$$= \frac{b}{h}(h - y) \, dy, \quad y = y \text{ from the } x \text{ axis.}$$

Integration. Applying the above equation and integrating with respect to y yields

$$\begin{aligned} \bar{y} &= \frac{\int_A \tilde{y} \, dA}{\int_A dA} = \frac{\int_0^h y \left[\frac{b}{h}(h - y) \, dy \right]}{\int_0^h \frac{b}{h}(h - y) \, dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh} \\ &= \frac{h}{3} \end{aligned}$$

Example 5.2.

Locate the centroid of the area shown in figure.



Solution

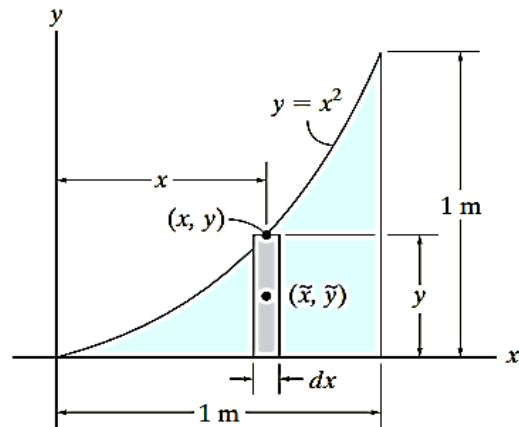
Differential Element. A differential element of thickness dx is shown in figure. The element intersects the curve at the arbitrary point (x, y) , and so it has a height y .

Area and Moment Arms. The area of the element is $dA = y \, dx$, and its centroid is located at $\tilde{x} = x, \tilde{y} = y/2$.

Integrations. Applying equations and integrating with respect to x yields

$$\bar{x} = \frac{\int_A \tilde{x} \, dA}{\int_A dA} = \frac{\int_0^{1 \text{ m}} xy \, dx}{\int_0^{1 \text{ m}} y \, dx} = \frac{\int_0^{1 \text{ m}} x^3 \, dx}{\int_0^{1 \text{ m}} x^2 \, dx} = \frac{0.250}{0.333} = 0.75 \text{ m}$$

$$\bar{y} = \frac{\int_A \tilde{y} \, dA}{\int_A dA} = \frac{\int_0^{1 \text{ m}} (y/2)y \, dx}{\int_0^{1 \text{ m}} y \, dx} = \frac{\int_0^{1 \text{ m}} (x^2/2)x^2 \, dx}{\int_0^{1 \text{ m}} x^2 \, dx} = \frac{0.100}{0.333} = 0.3 \text{ m}$$



Or

Differential Element. The differential element of thickness dy is shown in figure. The element intersects the curve at the arbitrary point (x, y) , and so it has a length $(1 - x)$.

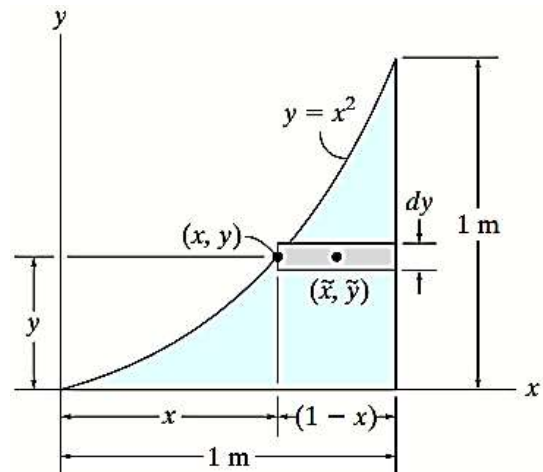
Area and Moment Arms. The area of the element is $dA = (1 - x) dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1 - x}{2} \right) = \frac{1 + x}{2}, \quad \tilde{y} = y$$

Integrations. Applying equations and integrating with respect to y , we obtain

$$\bar{x} = \frac{\int_A \tilde{x} dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} [(1 + x)/2](1 - x) dy}{\int_0^{1\text{ m}} (1 - x) dy} = \frac{\frac{1}{2} \int_0^{1\text{ m}} (1 - y) dy}{\int_0^{1\text{ m}} (1 - \sqrt{y}) dy} = \frac{0.250}{0.333} = 0.75 \text{ m}$$

$$\bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y(1 - x) dy}{\int_0^{1\text{ m}} (1 - x) dy} = \frac{\int_0^{1\text{ m}} (y - y^{3/2}) dy}{\int_0^{1\text{ m}} (1 - \sqrt{y}) dy} = \frac{0.100}{0.333} = 0.3 \text{ m}$$

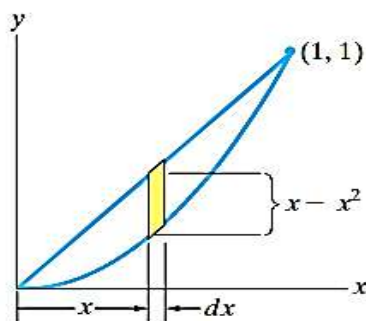
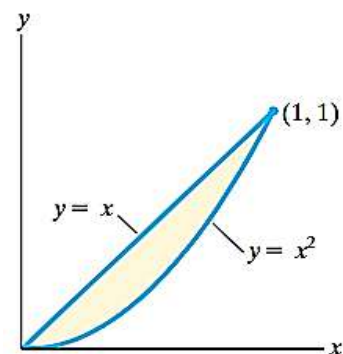


Example 5.3.

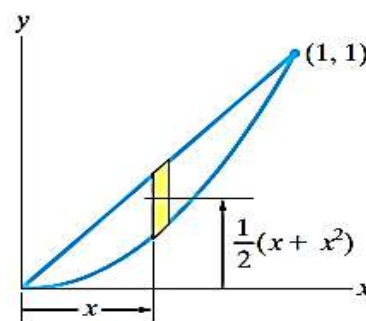
Determine the x and y positions of the centroid.

Solution

In this case the strip must be defined so that it extends from the lower curve ($y = x^2$) to the upper curve ($y = x$)



(a) A vertical strip of width dx . The height of the strip is equal to the difference in the two functions.



(b) The y coordinate of the midpoint of the strip.

Let dA be the vertical strip in Fig. a. The height of the strip is $x - x^2$ so $dA = (x - x^2) dx$. The x coordinate of the centroid is

$$\bar{x} = \frac{\int_A x dA}{\int_A dA} = \frac{\int_0^1 x(x - x^2) dx}{\int_0^1 (x - x^2) dx} = \frac{\left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1}{\left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1} = \frac{1}{2}.$$

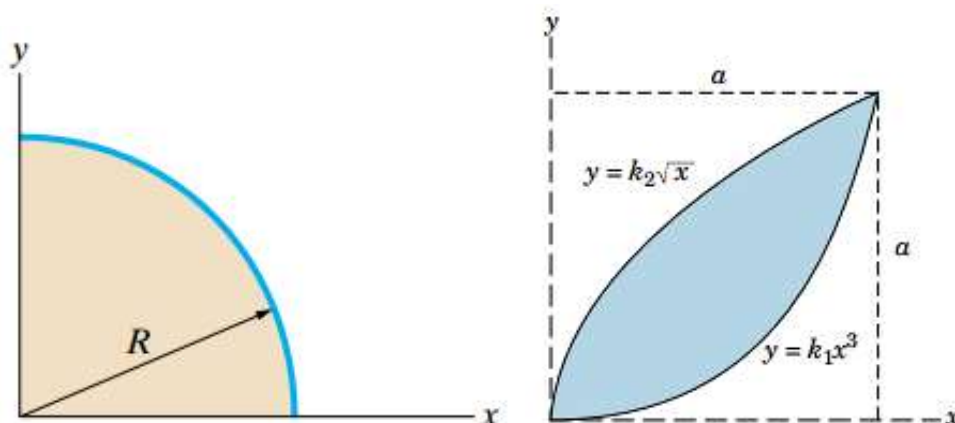
The y coordinate of the midpoint of the strip is $x^2 + \frac{1}{2}(x - x^2) = \frac{1}{2}(x + x^2)$ (Fig b). Substituting this expression for y in Equation, we obtain the y coordinate of the centroid.

$$\bar{y} = \frac{\int_A y dA}{\int_A dA} = \frac{\int_0^1 \left[\frac{1}{2}(x + x^2) \right] (x - x^2) dx}{\int_0^1 (x - x^2) dx} = \frac{\frac{1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1}{\left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1} = \frac{2}{5}.$$

Exercise 5.1

Locate the centroid for

- Quarter Circle
- Shaded area on the figure



5.5. Centroids of composite bodies

Although centroids of areas can be determined by integration, the process becomes difficult and tedious for complicated areas. In this section we describe a much easier approach that can be used if an area consists of a combination of simple areas, which we call a composite area. We can determine the centroid of a composite area without integration if the centroids of its parts are known.

Therefore the coordinates of the centroid of a composite area with an arbitrary number of parts are:

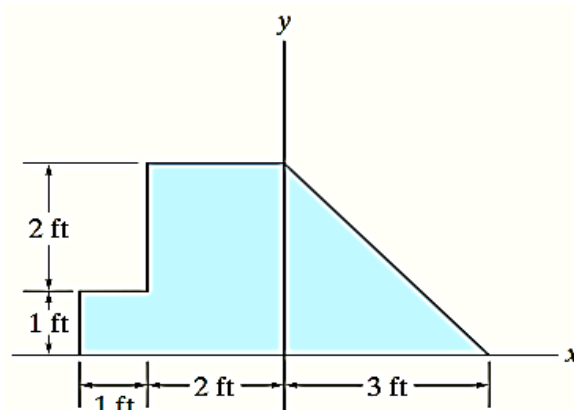
$$\bar{x} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}, \quad \bar{y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i}.$$

Procedures of determining the centroid of a composite area:

- Choose the parts Try to divide the composite area into parts whose centroids you know or can easily determine.
- Determine the values for the parts Determine the centroid and the area of each part. Watch for instances of symmetry that can simplify your task.
- Calculate the centroid using the above equations to determine the centroid of the composite area

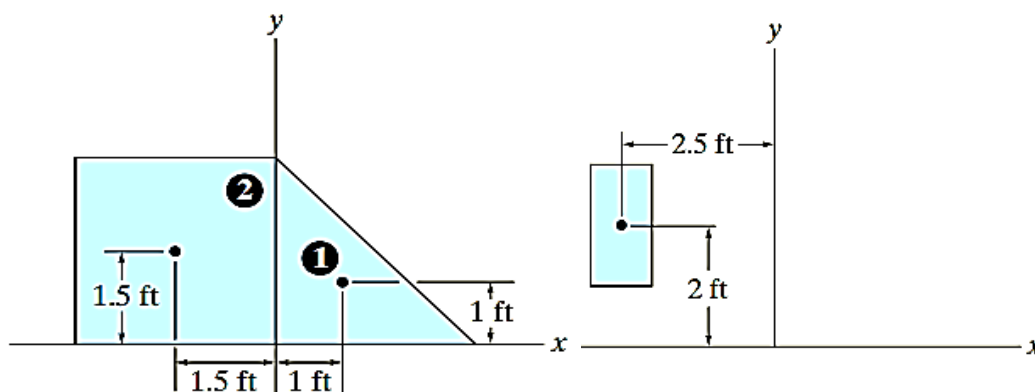
Example 5.4.

Locate the centroid of the plate area shown in figure.



Solution:

Composite Parts. The plate is divided into three segments as shown in figure. Here the area of the small rectangle 3 is considered “negative” since it must be subtracted from the larger one 2.



Moment Arms. The centroid of each segment is located as indicated in the figure. Note that the $\tilde{x}$ coordinates of 2 and 3 are negative.

Summations. Taking the data from above figures, the calculations are tabulated as follows:

Segment	$A \text{ (ft}^2\text{)}$	$\tilde{x} \text{ (ft)}$	$\tilde{y} \text{ (ft)}$	$\tilde{x}A \text{ (ft}^3\text{)}$	$\tilde{y}A \text{ (ft}^3\text{)}$
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	-1.5	1.5	-13.5	13.5
3	$-(2)(1) = -2$	-2.5	2	5	-4
	$\Sigma A = 11.5$			$\Sigma \tilde{x}A = -4$	$\Sigma \tilde{y}A = 14$

Thus,

$$\bar{x} = \frac{\sum \tilde{x}A}{\sum A} = \frac{-4}{11.5} = -0.348 \text{ ft}$$

$$\bar{y} = \frac{\sum \tilde{y}A}{\sum A} = \frac{14}{11.5} = 1.22 \text{ ft}$$

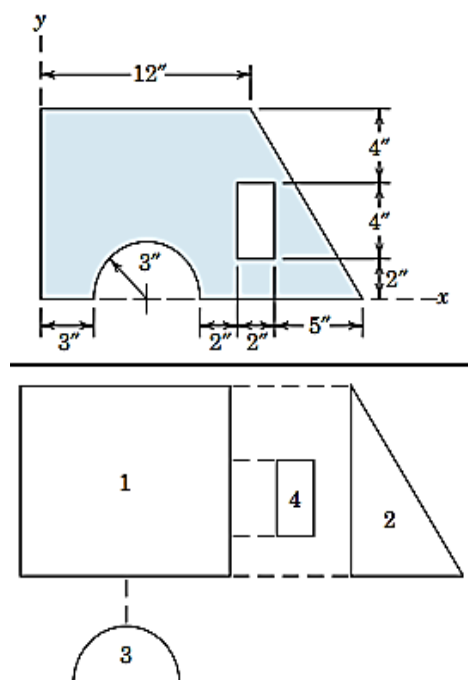
Example 5.5.

Locate the centroid of the shaded area.

Solution

The composite area is divided into the four elementary shapes shown in the figure. The centroid locations of all these shapes may be obtained from Table D/3. Note that the areas of the “holes” (parts 3 and 4) are taken as negative in the following table:

PART	A in. ²	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in. ³	$\bar{y}A$ in. ³
1	120	6	5	720	600
2	30	14	10/3	420	100
3	-14.14	6	1.273	-84.8	-18
4	-8	12	4	-96	-32
TOTALS	127.9			959	650



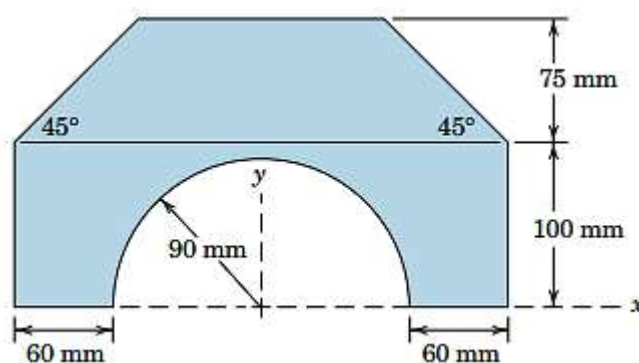
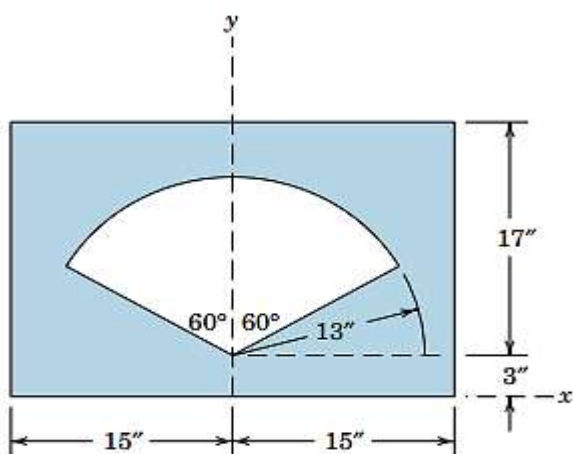
Thus,

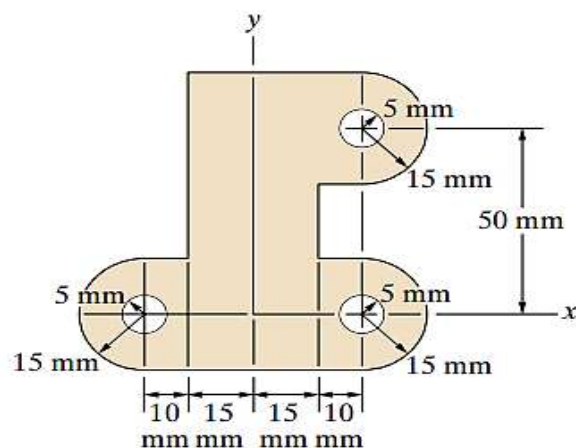
$$\left[\bar{X} = \frac{\sum A\bar{x}}{\sum A} \right] \quad \bar{X} = \frac{959}{127.9} = 7.50 \text{ in.}$$

$$\left[\bar{Y} = \frac{\sum A\bar{y}}{\sum A} \right] \quad \bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.}$$

Exercise 5.2.

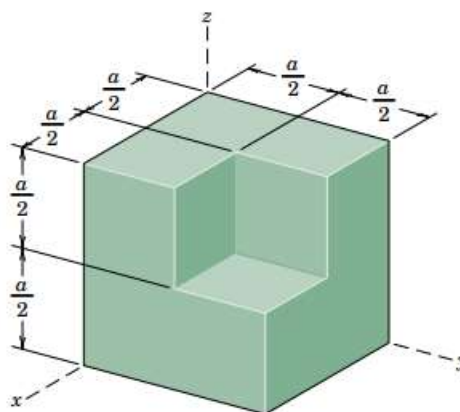
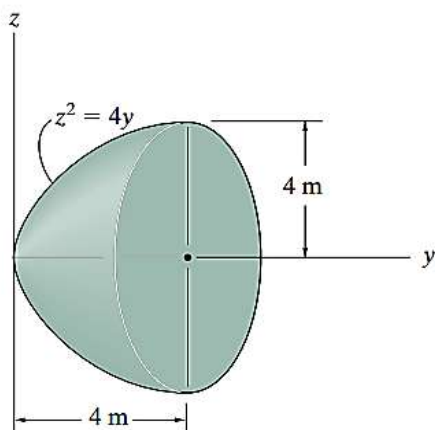
Determine the y-coordinate of the centroid of the following shaded areas.





Exercise 5.3.

Explain how to determine the centroid of solid objects and find the centroid of the following solid objects.



References

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